

Mie, Einstein and the Poynting-Robertson effect

J. Klačka

Faculty of Mathematics, Physics and Informatics, Comenius University
Mlynská dolina, 842 48 Bratislava, Slovak Republic
e-mail: klacka@fmph.uniba.sk

Abstract. A paradox associated with the astrophysical Poynting-Robertson effect is presented. The paradox arises when relativity theory and Mie's solution of Maxwell's equations are confronted with the statements on the Poynting-Robertson effect. Although the relevant physics has been known already for a century (Poynting 1903, Einstein 1905, Mie 1908), nobody has been aware of the inconsistency between the theories.

Key words. Poynting-Robertson effect, electromagnetic radiation, Mie's theory, relativity theory

1. Introduction

Physicists celebrate 100 years from the Mie's paper on the scattering of light by spherical particles (Mie 1908). Knowing the distribution of the material characteristics (refractive index and conductivity) within the particle, radius of the particle and wavelength(s) of the incoming light, Mie's (and Debye's; see also van de Hulst 1981, Bohren and Huffman 1983) calculation provides such physical quantities as cross sections of extinction, scattering and absorption of light. Moreover, since electromagnetic radiation generates a pressure force, Mie's solution of Maxwell's equations for electromagnetic field enables us to calculate cross section for radiation pressure.

Experimental evidence of the light pressure was presented by Russian physicist P. N. Lebedev at the end of the 19-th century. This result motivated another prominent physicist, J. M. Poynting, to formulate the problem of motion of a perfectly absorbing spherical particle under the action of incident light (Poynting 1903). Several different solutions were offered afterwards, until Robertson (1937) proposed a relativistically covariant equation of motion of the particle. Poynting and Robertson assumed that particle's "own radiation outwards being equal in all directions has zero resultant pressure" (Poynting 1903) in the particle's own frame of reference, or, "the process of absorption and re-emission

produces no net force on a particle when one chooses to work with a stationary frame referred to the particle” (Wyatt and Whipple 1950). The process of interaction of the incoming electromagnetic radiation with the spherical particle and the resulting motion of the particle is called the Poynting-Robertson (P-R) effect. It is being included on regular basis into the modelling of orbital evolution of cosmic dust grains under the action of electromagnetic radiation (e. g., Poynting 1903, Robertson 1937, Wyatt and Whipple 1950, Dohnanyi 1978, Jackson and Zook 1989, Gustafson 1994, Dermott *et al.* 1994, Reach *et al.* 1995, Quinn 2005, Grün 2007, Sykes 2007, Krügel 2008). The paper deals with an inconsistency between Mie’s solution of Maxwell’s equations, relativity theory and statements on the Poynting-Robertson effect.

2. Condition for the P-R effect: current status

Consider a spherical particle in its rest frame, and a beam of parallel photons striking upon it. If the flux density of radiation energy (energy flow through unit area perpendicular to the ray per unit time) is S and the geometric cross section of the particle is A ($\pi \times$ the radius of the particle squared), then the incident energy and momentum per unit time are

$$\begin{aligned} E_i &= S A , \\ \mathbf{p}_i &= \frac{E_i}{c} \mathbf{e} , \end{aligned} \tag{1}$$

where c is the speed of light and the unit vector \mathbf{e} defines the direction of the incoming photons (see also, e. g., pp. 5 and 11 in Burns *et al.* 1979). The condition for the P-R effect, as generally accepted, relates the total outgoing and incoming energies and momenta:

$$\begin{aligned} E_o &= E_i , \\ \mathbf{p}_o &= (1 - Q_{pr}) \mathbf{p}_i \end{aligned} \tag{2}$$

(see also, e. g., p. 5 and Eq. (10) on p. 10 in Burns *et al.* 1979), where Q_{pr} is the dimensionless efficiency factor for radiation pressure (defined as the ratio C_{pr}/A , where C_{pr} is cross section for the radiation pressure) given by optical properties of the spherical particle and wavelength(s) of the incoming radiation. The first of Eqs. (2) states that the outgoing and incoming energies are equal, which corresponds to the conservation of mass of the particle. The rates of change of energy and momentum of the particle due to the interaction with electromagnetic radiation are

$$\begin{aligned} \frac{d E}{d \tau} &= E_i - E_o = 0 , \\ \frac{d \mathbf{p}}{d \tau} &= \mathbf{p}_i - \mathbf{p}_o , \end{aligned} \tag{3}$$

where τ is the proper time measured in the frame of reference of the particle.

Eqs. (2) are in agreement with the statements published since the time of Poynting (1903). The case of perfectly absorbing particle corresponds to $Q_{pr} = 1$. Eqs. (2) yield $\mathbf{p}_o = 0$ for $Q_{pr} = 1$ and this is equivalent to the statement that the process of re-emission produces no net force on a particle in the proper frame of reference of the particle, provided re-emission is the only mechanism producing outgoing radiation (Poynting 1903, Robertson 1937, Wyatt and Whipple 1950, and subsequent papers and books).

3. Application of relativity theory

We know, according to Einstein and Minkowski, that from the quantities E_i , \mathbf{p}_i and E_o , \mathbf{p}_o we can compose two four-vectors

$$\begin{aligned} p_i^\mu &= \left(\frac{E_i}{c}, \mathbf{p}_i \right) = \left(\frac{E_i}{c}, \frac{E_i}{c} \mathbf{e} \right), \\ p_o^\mu &= \left(\frac{E_o}{c}, \mathbf{p}_o \right) = \left(\frac{E_i}{c}, \frac{E_i}{c} (1 - Q_{pr}) \mathbf{e} \right). \end{aligned} \quad (4)$$

In these equations also Eqs. (1) and (2) have been used. From the four-vectors p_i^μ and p_o^μ we can construct invariants M_i and M_o , with the physical meaning of the increase of mass of the incoming and outgoing radiation per unit time. We have

$$\begin{aligned} M_i &= \frac{1}{c} \sqrt{p_i^\mu p_{i\mu}} \equiv \frac{1}{c} \sqrt{\left(\frac{E_i}{c} \right)^2 - \mathbf{p}_i \cdot \mathbf{p}_i} = \\ &= \frac{E_i}{c^2} \sqrt{1 - \mathbf{e} \cdot \mathbf{e}} = 0 \end{aligned} \quad (5)$$

and

$$\begin{aligned} M_o &= \frac{1}{c} \sqrt{p_o^\mu p_{o\mu}} \equiv \frac{1}{c} \sqrt{\left(\frac{E_o}{c} \right)^2 - \mathbf{p}_o \cdot \mathbf{p}_o} = \\ &= \frac{E_i}{c^2} \sqrt{1 - (1 - Q_{pr})^2}. \end{aligned} \quad (6)$$

Eq. (6) yields a nontrivial result: $1 - (1 - Q_{pr})^2 = Q_{pr} (2 - Q_{pr}) \geq 0$, or,

$$0 \leq Q_{pr} \leq 2. \quad (7)$$

4. Application of Mie's theory

Mie's theory is a solution of Maxwell's equations for electricity and magnetism. The solution holds for the case when an incoming electromagnetic radiation interacts with a spherical particle. Mie's theory offers also the values of Q_{pr} , the fundamental quantity for the P-R effect. What are the values of Q_{pr} ? For our purposes it is important that they can be larger than 2 [e. g., van de Hulst (1981) presents also values larger than 2.5 in Table 13 on p. 161].

5. Inconsistency in physics

We have got a problem. Relativity theory states that $0 \leq Q_{pr} \leq 2$ (see Eq. 7), but Mie's theory offers also values $Q_{pr} > 2$!

Relativity theory was motivated by Maxwell's equations. However, Mie's theory is a special solution of the Maxwell's equations. Why an inconsistency exists between these two approaches? Why the inconsistency exists 100 years?

The conclusion is that physicists and astrophysicists sometimes use also theories which yield inconsistent results. But instead of being disappointed, an optimistic view is possible: "How wonderful that we have met with paradox. Now we have some hope of making progress." (Niels Bohr).

6. Fundamental condition for the P-R effect: correct statement

In order to solve the paradox, we have to realize that the statement "the process of re-emission produces no net force on a perfectly absorbing spherical particle when one chooses to work with a stationary frame referred to the particle" ($Q_{pr} = 1$ in Eq. 2) concerns only a part of the outgoing radiation. In the complete description, one has to take into account also diffraction (small angle scattering) of the light. As a matter of fact, there exists "extinction paradox" (van de Hulst 1981, p. 107) according to which diffracted light plays a non-negligible role in treating the incoming and outgoing radiation, even for particles that are large in comparison with the wavelength of the interacting light; thus, diffraction must be included into considerations even in the situation in which one normally gets along with the geometrical optics approximation. We know that the diffracted light gives a zero contribution to the radiation pressure of large spheres (van de Hulst 1981, p. 225), but diffraction cannot be neglected in a separate treatment of the incoming and outgoing radiation.

The correct physics gives the following result for a large perfectly absorbing spherical particle: the dimensionless efficiency factors for absorption and scattering are $Q_{abs} = 1$ (this result is used in the P-R effect: $E_i = SAQ_{abs} = SA$ in Eq. (1)) and $Q_{sca} = 1$ (due to the diffraction). These two factors sum up into the efficiency factor of extinction $Q_{ext} = Q_{abs} + Q_{sca}$ (cross sections can be obtained by multiplication with geometrical cross section A). The correct result for the large perfectly absorbing spherical particle, with the effect of diffraction taken into account, is $Q_{ext} = 2$. The effect of diffraction is equally important as the effect of absorption. As a consequence, the conventional condition for the P-R effect $E_i = SA$ (Eq. (1)) has to be replaced by the physical condition $E_i = SAQ_{ext}$. Thus, we have the following condition for the incoming radiation

$$\begin{aligned} E_i &= S A Q_{ext} , \\ \mathbf{p}_i &= \frac{E_i}{c} \mathbf{e}, \end{aligned} \tag{8}$$

instead of Eq. (1). Moreover, the second condition for the P-R effect cannot be of the form (2). It can be easily seen from Eqs. (1)-(3) and Eq. (8) that Eqs. (3) do not change

if

$$\begin{aligned} E_o &= E_i, \\ \mathbf{p}_o &= \left(1 - \frac{Q_{pr}}{Q_{ext}}\right) \mathbf{p}_i. \end{aligned} \quad (9)$$

Eqs. (8) and (9) are the conditions under which the P-R effect holds. The conditions are formulated in the proper frame of reference of the particle. The case of a large perfectly absorbing sphere corresponds to $\mathbf{p}_o = \mathbf{p}_i / 2$ and not to $\mathbf{p}_o = 0$ as it has been conventionally stated.

The expression for the four-vector p_o^μ ,

$$p_o^\mu = \left(\frac{E_o}{c}, \mathbf{p}_o\right) = \left(\frac{E_i}{c}, \frac{E_i}{c} \left(1 - \frac{Q_{pr}}{Q_{ext}}\right) \mathbf{e}\right), \quad (10)$$

yields for the increase of mass of the outgoing radiation per unit time

$$\begin{aligned} M_o &= \frac{1}{c} \sqrt{p_o^\mu p_{o\mu}} \\ &= \frac{E_i}{c^2} \sqrt{1 - \left(1 - \frac{Q_{pr}}{Q_{ext}}\right)^2}. \end{aligned} \quad (11)$$

The non-negativity of the expression under the square root symbol is equivalent to

$$0 \leq \frac{Q_{pr}}{Q_{ext}} \leq 2. \quad (12)$$

Physical condition represented by Eq. (12) differs from the condition presented in Eq. (7) and is consistent with Mie's solution of Maxwell's equations.

7. Conclusion

A paradox coming from the application of relativity theory and Mie's solution of Maxwell's equations to the Poynting-Robertson effect is presented. Solution of the paradox resides in the fact that diffraction plays a non-negligible role in the process of interaction between the incoming radiation and the spherical particle. As a consequence, in the case of a large perfectly absorbing spherical particle there holds the condition $\mathbf{p}_o = \mathbf{p}_i / 2$ for the outgoing and incoming momenta per unit time, and not the conventional condition $\mathbf{p}_o = 0$ obtained by the neglect of diffraction. Physically correct relations defining the P-R effect are given by Eqs. (8) and (9).

Acknowledgements. The author is indebted to V. Balek for his valuable comments. This work was supported by the Scientific Grant Agency VEGA, Slovakia, grant No. 1/3074/06.

References

- Bohren C. F., Huffman D. R., 1983. *Absorption and Scattering of Light by Small Particles*, John Wiley & Sons, Inc., New York.
- Burns J. A., Lamy P. L., Soter S., 1979. Radiation forces on small particles in the Solar System. *Icarus* **40**, 1-48.

- Dermott S. F., Jayaraman S., Xu Y. L., Gustafson B. A. S., Liou J. C., 1994. A circumsolar ring of asteroidal dust in resonant lock with the Earth. *Nature* **369**, 719-723.
- Dohnanyi J. S., 1978. Particle dynamics. In: Cosmic Dust, J. A. M. McDonnell (Ed.), Wiley-Interscience, Chichester, 527-605.
- Einstein A., 1905. Zur Elektrodynamik der bewegter Körper. *Annalen der Physik* **17**, 891-920.
- Gustafson, B. A. S., 1994. Physics of Zodiacal Dust. *Annual Review of Earth and Planetary Sciences* **22**, 553-595.
- Grün E., 2007. Solar System dust. In: Encyclopedia of the Solar System, L.-A. McFadden, P. R. Weissmann and T. V. Johnson (eds.), Academic Press (Elsevier), San Diego, 2nd ed., 621-636.
- Jackson A. A., Zook H. A., 1989. A Solar System dust ring with the Earth as its shepherd. *Nature* **337**, 629-631.
- Krügel E., 2008. An introduction to the physics of interstellar dust. Taylor & Francis, Boca Raton, 387pp.
- Mie G., 1908. Beiträge zur Optik trüber Medien speziell kolloidaler Metalösungen. *Ann. Phys.* **25**, 377-445.
- Poynting J. M., 1903. Radiation in the Solar System: its Effect on Temperature and its Pressure on Small Bodies. *Philosophical Transactions of the Royal Society of London Series A* **202**, 525-552.
- Quinn, T., 2005. Planet Formation. In: Chaos and Stability in Planetary Systems, Dvorak R., Freistetter F., Kurths J. (eds.), Springer-Verlag, Berlin, 187-217.
- Reach W. T., Franz B. A., Welland J. L., Hauser M. G., Kelsall T. N., Wright E. L., Rawley G., Stemwedel S. W., Splesman W. J., 1995. Observational confirmation of a circumsolar dust ring by the COBE satellite. *Nature* **374**, 521-523.
- Robertson H. P., 1937. Dynamical effects of radiation in the Solar System. *Mon. Not. R. Astron. Soc.* **97**, 423-438.
- Sykes M. V., 2007. Infrared views of the Solar System from space. In: Encyclopedia of the Solar System, L.-A. McFadden, P. R. Weissmann and T. V. Johnson (eds.), Academic Press (Elsevier), San Diego, 2nd ed., 681-694.
- van de Hulst H. C., 1981. Light Scattering by Small Particles. Dover Publications, Inc. New York, 470 pp. (originally published in 1957 by John Wiley & Sons, Inc., New York)
- Wyatt S. P., Whipple F. L., 1950. The Poynting-Robertson effect on meteor orbits. *Astrophys. J.* **111**, 558-565.